

Deformation quantization of the Pais–Uhlenbeck fourth order oscillator

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Abstract

We analyze the quantization of the Pais–Uhlenbeck fourth order oscillator within the framework of deformation quantization. Our approach exploits the Noether symmetries of the system by proposing integrals of motion as the variables to obtain a solution to the \star -genvalue equation, namely the Wigner function. We also obtain, by means of a quantum canonical transformation the wave function associated to the Schrödinger equation of the system. We show that unitary evolution of the system is guaranteed by means of the quantum canonical transformation and via the properties of the constructed Wigner function, even in the so called equal frequency limit of the model, in agreement with recent results.

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1. Introduction

Whenever one consider curvature terms, for example in general relativity or brane inspired models, one is faced naturally with field theories described by Lagrangians with higher order derivative terms. The Pais–Uhlenbeck

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fourth order linear oscillator, originally introduced in [1], is perhaps the simplest example and definitely the best known higher derivative mechanical system and, in particular, it has served as a toy model to understand several important issues related to Ostrogradsky instabilities emerging naturally in higher order field theories [2, 3, 4, 5, 6, 7, 8]. Recently, the Pais–Uhlenbeck oscillator has been used as a guide to study higher order structures associated to supersymmetric field theory [9], PT -symmetric Hamiltonian mechanics [10], and geometric models within the scalar field cosmology context [11]. In this sense, it is important to mention that naive quantization procedures for the Pais–Uhlenbeck model has to be enhanced in order to recover unitarity in a physical allowed sector. Our main motivation is thus to address for the Pais–Uhlenbeck oscillator, within the perspective of the deformation quantization formalism, the long standing problems associated to the non-unitarity of higher derivative theories. As we will see, within our formulation the unitarity is guaranteed straightforwardly, even in the equal frequency limit of the model, by the introduction of a well-defined Wigner distribution.

The framework of deformation quantization was introduced in [12] as an alternative approach to the problem of quantization. In this formalism one uses, as guidelines, the Dirac quantization rules in order to pass from classical physics to the quantum realm. As is to be expected, a consistency requirement for such a quantum theory is the existence of a classical limit, that is, a quantum system should reduce to its classical counterpart whenever the limit of \hbar , the Planck constant, tends to zero. From this perspective, the quantization of a classical system could be seen as a deformation of the algebraic structures involved in a parameter encoding the quantum nature associated to the system (\hbar in our case). Furthermore, the quantization rules require that for any classical observable there is a corresponding quantum observable, and similarly, that the Poisson bracket corresponds to the quantum commutator. All these requirements can be achieved by replacing the usual product of the algebra of smooth functions on the classical phase space with an associative non-commutative product, depending on \hbar , such that the resulting commutator is a deformation of the Poisson bracket. We refer the reader to [13], where results on the explicit construction of maps between classical and quantum observables are explained in detail, to Refs. [14, 15], where conditions on the existence of the star product are exposed, and to the reviews [16, 17, 18] for general aspects of deformation quantization, as well as for more recent developments.

Our approach is based on taking advantage of the symmetries inherent to

the Pais–Uhlenbeck model in order to construct the Wigner function that contains the relevant quantum information of the system. In this manner, we show that there exists a couple of integrals of motion associated to Noether charges, which in turn serve as privileged variables in order to find the solutions to the \star -genvalue equation. Further, in order to obtain the quantum wave functions we consider both, classical and quantum canonical transformations. At a classical level we transform, in a standard way, the Pais–Uhlenbeck system to a simpler model composed of the difference of two uncoupled harmonic oscillators for which the Wigner function may be also obtained. We then use the latter Wigner function to obtain a wave equation which, by means of a quantum canonical transformation, may be used in order to obtain the quantum wave function for the original Pais–Uhlenbeck system. The resulting wave function is identical to the one obtained in [6] by a different reasoning. We also show that in the equal frequency limit of the model the source of the continuous spectrum can be traced out through a linear canonical transformation that maps the Pais–Uhlenbeck Hamiltonian to a Hamiltonian composed of a discrete spectrum part plus a continuous spectrum part, contrary to the unequal frequency case. Besides, we demonstrate that in the equal frequency limit the Wigner function is certainly unitary as consequence of composition of unitary transformations considered through the quantum canonical transformations. In this sense, our results explicitly manifest the ghost-free feature of the Pais–Uhlenbeck model, in complete agreement with [4, 6].

The article is organized as follows. In Section 2, we include a brief review of deformation quantization in order to set our notation and to define some useful structures. We also consider quantum canonical transformations as they will be essential in our context to obtain the wave functions associated to the Pais–Uhlenbeck oscillator. In Section 3, we analyze the Wigner function for our model in terms of its integrals of motion and we identify the quantum wave equation. Also, in this section we detail the equal frequency limit for the Pais–Uhlenbeck oscillator. In Section 4, we include some concluding remarks. Finally, we address some technical issues related to the construction of the Pais–Uhlenbeck wave function in the Appendix.

2. Deformation quantization

2.1. Basic notions

The origins of deformation quantization were first introduced by H. Weyl in [19]. The main idea is to associate a quantum operator $\hat{W}[f]$ in the Hilbert space $L^2(\mathbb{R}^n)$ to every classical observable $f(q, p)$ defined on the phase space \mathbb{R}^{2n} . This operator is known as the Weyl operator and it is explicitly given by

$$f(q, p) \mapsto \hat{W}[f] := \frac{1}{(2\pi\hbar)^{2n}} \int_{\mathbb{R}^{2n}} d^n\eta \, d^n\xi \exp \left[\frac{i}{\hbar} (\hat{q} \cdot \eta + \hat{p} \cdot \xi) \right] \tilde{f}(\eta, \xi), \quad (1)$$

where \tilde{f} is the Fourier transform of $f \in L^2(\mathbb{R}^{2n})$ given by

$$\tilde{f}(\eta, \xi) = \int_{\mathbb{R}^{2n}} d^nq \, d^np \exp \left[-\frac{i}{\hbar} (\eta \cdot q + \xi \cdot p) \right] f(q, p), \quad (2)$$

\hat{q} , \hat{p} are operators satisfying the canonical commutations relations, and the integral is understood in the weak operator topology [20, 21]. Later on, Wigner [22] obtained an inverse formula which maps a quantum operator into its symbol, that is, a differential operator with polynomial coefficients defined on classical phase space. This map, known as the Wigner function, results a quasi-probability distribution function, explicitly written as

$$\rho(q, p) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} d^ny \, \psi^* \left(q - \frac{\hbar}{2}y \right) e^{-iy \cdot p} \psi \left(q + \frac{\hbar}{2}y \right). \quad (3)$$

Among the most important properties of the Wigner function we would like to point out the following. In the first place, Wigner function is a particular representation of the density matrix which is normalized, bounded and real. In the second place, for a given quantum wave function $\psi(x)$, Wigner function represents a generating function for all spatial auto-correlations. Finally, in the classical limit $\hbar \mapsto 0$ Wigner function reduces to a highly localized probability density in the coordinate space [23].

Subsequently, Moyal found an explicit formula for the symbol of a quantum commutator between operators [24],

$$(f \star g)(q, p) = f \exp \left(\frac{i\hbar}{2} \overleftarrow{\partial}_{q^i} \overrightarrow{\partial}_{p_i} - \frac{i\hbar}{2} \overleftarrow{\partial}_{p_i} \overrightarrow{\partial}_{q^i} \right) g. \quad (4)$$

This \star -product is known as the Moyal product, and will be defined as an involution in (7). For a two dimensional case ($n = 1$) this deformed product reads

$$f \star g = \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{i\hbar}{2} \right)^k \sum_{m=0}^k \binom{k}{m} (-1)^m (\partial_q^{k-m} \partial_p^m f) (\partial_q^m \partial_p^{k-m} g). \quad (5)$$

Straightforwardly, one may check that the Moyal product satisfies the following properties:

1. $f \star g = fg + \mathcal{O}(\hbar)$, that is, the \star -product is a deformation of the usual pointwise product between smooth functions.
2. With respect to the \star -product, the Weyl operator \hat{W} is a homomorphism of algebras,

$$\hat{W} : (C^\infty(\mathbb{R}^{2n}), \star) \rightarrow (\mathcal{L}(L^2(\mathbb{R}^n)), \circ), \quad (6)$$

where

$$\hat{W}[f \star g] := \hat{W}[f] \circ \hat{W}[g], \quad (7)$$

between distributional functions defined on the phase space and the space of bounded operators on the square integrable functions $L^2(\mathbb{R}^n)$ under the composition \circ as a product.

3. The deformed Poisson bracket associated to the \star -product is given by

$$[f, g]_\star := \frac{1}{i\hbar} (f \star g - g \star f) = \{f, g\} + \mathcal{O}(\hbar), \quad (8)$$

where $\{\cdot, \cdot\}$ denotes the standard Poisson bracket, and $[\cdot, \cdot]_\star$ is known as the Moyal commutator or the Moyal bracket. As a consequence of (??), Moyal bracket may be interpreted as a Lie bracket on the space of Weyl operators [25].

In order to avoid convergence problems, the Moyal product is defined not in the space of smooth functions, $C^\infty(\mathbb{R}^{2n})$, but in the extended space, $C^\infty(\mathbb{R}^{2n})[[\hbar]]$, which corresponds to the space of formal power series in \hbar with coefficients in $C^\infty(\mathbb{R}^{2n})$, or alternatively, it can be defined on the space of Schwartz functions using an integral representation and then extend it to a suitable space of distributions [26, 27]. The interpretation of this product as a non-commutative deformation on the algebra of observables was introduced

in [12] by means of Gerstenhaber's algebras. In particular, the existence of a \star -product for an arbitrary symplectic manifold was demonstrated in [15]. Subsequently, as a consequence of the formality theorem, Kontsevich solved the existence and classification problem of star products on a generic finite dimensional Poisson manifold [28].

2.2. Time evolution and canonical transformations

The expectation values of observables and the time evolution of states can be computed in a similar manner as in classical mechanics, however, the usual pointwise product between functions and the Poisson bracket are replaced with the \star -product and the Moyal commutator, respectively. In consequence, the expectation value of an observable A in a state ψ , is given by the expression

$$\langle A \rangle_\psi = \frac{1}{(2\pi\hbar)^{n/2}} \int_{\mathbb{R}^{2n}} d^n q d^n p A(q, p) \star \rho(q, p), \quad (9)$$

where $\rho(x, p)$ is the Wigner function defined in (3). In this sense, expectation values of physical observables in deformation quantization correspond to obtaining the trace of an operator with the density matrix, in close analogy to standard probability theory [23].

The dynamical equation of the quantum distribution $\rho(x, p)$, results in the counterpart of Liouville's theorem in classical mechanics describing the time evolution of a classical distribution function, and is given by the formula

$$\frac{\partial \rho(q, p)}{\partial t} = \frac{1}{i\hbar} [H, \rho]_\star = \frac{H \star \rho - \rho \star H}{i\hbar}, \quad (10)$$

where H is a distinguished real function from the algebra of observables, namely, the Hamiltonian. Being time-dependent, this evolution equation, also known as Moyal's equation, does not completely determine the Wigner function for a system [29]. Then, just as in the conventional formulation of quantum mechanics, a systematic solution may be inferred from the spectrum of the stationary problem. Static Wigner functions obey a more suggestive \star -genvalue equation, inducing Bopp shifts [16]

$$\begin{aligned} H(q, p) \star \rho(q, p) &= \rho(q, p) \star H(q, p) = H \left(q + \frac{i\hbar}{2} \partial_p, p - \frac{i\hbar}{2} \partial_q \right) \rho(q, p) \\ &= E \rho(q, p), \end{aligned} \quad (11)$$

where E corresponds to the energy eigenvalue associated to the Hamiltonian, leading to the spectral properties of the Wigner function as a quantum distribution function [16]. These quantum properties are related to the fact that Wigner function is not positive semi-definite, allowing in principle negative values for certain areas of the phase space. This counter intuitive negative-probability aspect has been speculated as a way to detect quantum interference which may be measured and reconstructed indirectly in the laboratory [30].

As within the framework of deformation quantization the observables are represented by smooth functions defined on phase space, and thus they transform classically, the outcome of a canonical transformation on the quantum \star -genvalue equations results in an appropriate transformation of the Wigner function [31].

A general transformation of the phase space coordinates is defined as a smooth bijective map $T : \mathbb{R}^{2n} \ni (q, p) \rightarrow (Q, P) \in \mathbb{R}^{2n}$, which transforms every observable $A \in C^\infty(\mathbb{R}^{2n})$ by

$$A' = A \circ T. \quad (12)$$

Further, the star product in the new variables \star' satisfies the natural condition [32]

$$(f \star g) \circ T = (f \circ T) \star' (g \circ T), \quad f, g \in C^\infty(\mathbb{R}^{2n}), \quad (13)$$

in such a way that the \star' -product is given by

$$f \star' g = f \exp \left(\frac{i\hbar}{2} \overleftarrow{D}_{Q^i} \overrightarrow{D}_{P_i} - \frac{i\hbar}{2} \overleftarrow{D}_{P_i} \overrightarrow{D}_{Q^i} \right) g, \quad (14)$$

where the vector fields D_{Q^i} , D_{P_i} correspond to the transformed derivations ∂_{q^i} , ∂_{p_i} according to the rule stated in equation (12).

Among the phase space coordinate maps there are some which play a special role in classical mechanics, namely the canonical transformations or symplectomorphisms, that is, those maps which preserve the symplectic form, and thus the Poisson bracket structure

$$\{q^i, p_j\} = \{Q^i, P_j\} = \delta_j^i. \quad (15)$$

In phase space quantum mechanics, a quantum canonical transformation is a transformation T such that it preserves the form of the corresponding deformed Poisson bracket, namely, the Moyal's bracket

$$[q^i, p_j]_\star = [Q^i, P_j]_{\star'} = \delta_j^i, \quad (16)$$

where $[\cdot, \cdot]_{\star'}$ denotes the Moyal's bracket transformed by T to the new coordinate system [32, 33],

$$[f, g]_{\star'} = [f \circ T^{-1}, g \circ T^{-1}]_{\star} \circ T. \quad (17)$$

The transformation of coordinates T induces a unitary operator on the Hilbert space $\hat{U}_T : L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)$, transforming vector states and observables to the new coordinate system in such a way that for any quantum observable \hat{A} the transformation reads

$$\hat{A}'(\hat{Q}, \hat{P}) = \hat{U}_T \hat{A}(\hat{q}, \hat{p}) \hat{U}_T^{-1}. \quad (18)$$

This expression allows us to derive the form of the operator \hat{U}_T , which corresponds to the celebrated Dirac's quantum transformation [34]

$$(\hat{U}_T \psi)(Q) = \frac{1}{(2\pi\hbar)^{n/2}} \int d^n q \sqrt{\left| \frac{\partial^2 F}{\partial q \partial Q}(q, Q) \right|} e^{\frac{i}{\hbar} F(q, Q)} \psi(q). \quad (19)$$

being $F(q, Q)$ the generator of the classical canonical transformation such that

$$p = \frac{\partial F(q, Q)}{\partial q}, \quad P = -\frac{\partial F(q, Q)}{\partial Q}. \quad (20)$$

In this expression, the implementation of the quantum canonical transformation on state eigenfunctions is realized through a generalization of a Fourier-type transformation containing \hbar -corrections with respect to the classical canonical counterpart, thus encompassing the quantum nature of the operator. In addition, it can be demonstrated that \hat{U}_T is indeed a unitary operator on the Hilbert space $L^2(\mathbb{R}^n)$ [31, 32].

3. Deformation quantization of the Pais–Uhlenbeck Oscillator

In this section, we perform a deformation quantization of the Pais–Uhlenbeck oscillator by solving the corresponding \star -genvalue equations and, in order to obtain the associated wave functions, we make use of the theory of quantum canonical transformations discussed in the previous section.

3.1. Wigner function

In order to study the quantum dynamics of the system, we will start by considering the Lagrangian

$$L(q, \dot{q}, \ddot{q}) = \frac{1}{2} [\ddot{q}^2 - (\Omega_1^2 + \Omega_2^2)\dot{q}^2 + \Omega_1^2\Omega_2^2q^2] , \quad (21)$$

where q is a real-valued function of time, \dot{q} and \ddot{q} its first and second time-derivatives, respectively, and the parameters Ω_1 and Ω_2 correspond to a pair of frequencies which are taken real and positive. More explicitly, this model is characterized by the following fourth-order differential equation of motion

$$\frac{d^4q}{dt^4} + (\Omega_1^2 + \Omega_2^2)\frac{d^2q}{dt^2} + \Omega_1^2\Omega_2^2q = 0 , \quad (22)$$

which can be derived directly from the Lagrangian. The canonical Hamiltonian can be obtained by means of the Ostrogradsky's method [35]. To implement this method, the phase space involves, in addition to the canonical coordinates (q, p_q) , an extra canonical pair of variables, namely $x := \dot{q}$ with corresponding canonical momentum p_x [5, 6, 8]. The momenta are defined as $p_x := \partial L / \partial \dot{q}$ and $p_q := \partial L / \partial \ddot{q} - d/dt (\partial L / \partial \dot{q})$, and in our case they explicitly read

$$\begin{aligned} p_x &= \ddot{q} , \\ p_q &= -(\Omega_1^2 + \Omega_2^2)\dot{q} - \frac{d^3q}{dt^3} , \end{aligned} \quad (23)$$

respectively, while the canonical Hamiltonian is obtained through the Legendre transformation $H(q, p_q, x, p_x) = p_q\dot{q} + p_x\dot{x} - L(q, \dot{q}, \ddot{q})$, and thus

$$H(q, p_q, x, p_x) = p_qx + \frac{p_x^2}{2} + \frac{(\Omega_1^2 + \Omega_2^2)x^2}{2} - \frac{\Omega_1^2\Omega_2^2q^2}{2} . \quad (24)$$

For this theory, it is natural to define the generalized Poisson bracket

$$\{A, B\} := \frac{\partial A}{\partial q} \frac{\partial B}{\partial p_q} - \frac{\partial A}{\partial p_q} \frac{\partial B}{\partial q} + \frac{\partial A}{\partial x} \frac{\partial B}{\partial p_x} - \frac{\partial A}{\partial p_x} \frac{\partial B}{\partial x} , \quad (25)$$

in order to obtain canonical relations

$$\{q, p_q\} = 1 = \{x, p_x\} , \quad (26)$$

while the rest of the brackets among the phase space variables are vanishing. This Poisson structure enable us to write the canonical Hamilton equations of motion which, by making the identification $x = \dot{q}$, straightforwardly lead to equation (22). Therefore, the Hamiltonian H , obtained in (24), describes the dynamics in phase space for the fourth order Pais–Uhlenbeck model, and thus this Hamiltonian corresponds to the one which is to be quantized, using in our case, the techniques related to deformation quantization as reviewed above.

Our goal now is to calculate the Wigner function. As we will see below, the \star -genvalue problem (11) can be solved directly without first solving the corresponding Schrödinger equation. Considering the Pais–Uhlenbeck oscillator with Hamiltonian given by (24), the \star -genvalue equation (11) for the Wigner function $\rho(q, p_q, x, p_x)$ explicitly reads

$$\left[\left(p_q - \frac{i\hbar}{2} \partial_q \right) \left(x + \frac{i\hbar}{2} \partial_{p_x} \right) + \left(\frac{\Omega_1^2 + \Omega_2^2}{2} \right) \left(x + \frac{i\hbar}{2} \partial_{p_x} \right)^2 + \frac{1}{2} \left(p_x - \frac{i\hbar}{2} \partial_x \right)^2 - \frac{\Omega_1^2 \Omega_2^2}{2} \left(q + \frac{i\hbar}{2} \partial_{p_q} \right)^2 \right] \rho = E \rho, \quad (27)$$

where we have used the associative \star -product

$$\star := \exp \left[\frac{i\hbar}{2} \left(\overleftarrow{\partial}_q \overrightarrow{\partial}_{p_q} - \overleftarrow{\partial}_{p_q} \overrightarrow{\partial}_q + \overleftarrow{\partial}_x \overrightarrow{\partial}_{p_x} - \overleftarrow{\partial}_{p_x} \overrightarrow{\partial}_x \right) \right], \quad (28)$$

which is naturally associated to the canonical Poisson structure (25) of the Pais–Uhlenbeck oscillator. The eigenvalue equation (27) may be decomposed into two partial differential equations corresponding to the real and imaginary parts of the equation, respectively. These equations explicitly read

$$\left[p_q x + \frac{p_x^2}{2} + \frac{\hbar^2}{4} \left(\partial_q \partial_{p_x} - \frac{1}{2} \partial_x^2 \right) - \frac{\Omega_1^2 + \Omega_2^2}{2} \left(x^2 - \frac{\hbar^2}{4} \partial_{p_x}^2 \right) - \frac{\Omega_1^2 \Omega_2^2}{2} \left(q^2 - \frac{\hbar^2}{4} \partial_{p_q}^2 \right) - E \right] \rho(q, p_q, x, p_x) = 0, \quad (29)$$

for the real part, while

$$\left[p_q \partial_{p_x} - x \partial_q - p_x \partial_x + (\Omega_1^2 + \Omega_2^2) x \partial_{p_x} - \Omega_1^2 \Omega_2^2 q \partial_{p_q} \right] \rho(q, p_q, x, p_x) = 0, \quad (30)$$

corresponds to the imaginary part. Although, these equations may seem challenging at first sight, we can take advantage of the deeply relation between Moyal quantization and the canonical structure of the theory in order to find solutions to these equations. By using the symmetries of the Hamiltonian, and employing the conserved Noether charges as variables for the Wigner function, one can infer the solution to the \star -genvalue problem [36]. Indeed, using Noether theorem, suitable generalized to higher derivative theories, one finds that the Pais–Uhlenbeck oscillator remains invariant under the symmetry [37],

$$q \mapsto q + \varepsilon \left(\frac{d^3 q}{dt^3} \pm (\Omega_1^2 - \Omega_2^2) \frac{dq}{dt} \right), \quad (31)$$

which, as a consequence, imply the existence of two global integrals of motion

$$J_1 = \frac{1}{\Omega_1^2 - \Omega_2^2} \left[\Omega_1^2 (p_x + \Omega_2^2 q)^2 + (p_q + \Omega_1^2 x)^2 \right], \quad (32)$$

$$J_2 = \frac{1}{\Omega_1^2 - \Omega_2^2} \left[(p_q + \Omega_2^2 x)^2 + \Omega_2^2 (p_x + \Omega_1^2 q)^2 \right]. \quad (33)$$

These integrals of motion are associated to infinitesimal transformations of the form (31). However, when one considers infinitesimal transformations along the velocity vector field, \dot{q} , together with infinitesimal time transformations, the integrals of motion are associated to the so-called energies for a second order differential Lagrangian. In particular, one may check that the Pais–Uhlenbeck Hamiltonian (24) is related to the integrals of motion of our interest by the identity $H = (J_1 - J_2)/2$. A general statement of Noether theorem and the energies for a second order Lagrangian system is reviewed in [38, Appendix A], while a detailed exposition may be found in [39].

Thus, using constants of motion (32) and (33), the Wigner function given by

$$\rho_{nm}(q, p_q, x, p_x) = \frac{(-1)^{m+n}}{\pi^2 \hbar^2} e^{-2J_1/\hbar\Omega_1} e^{-2J_2/\hbar\Omega_2} L_n \left(\frac{4J_1}{\hbar\Omega_1} \right) L_m \left(\frac{4J_2}{\hbar\Omega_2} \right), \quad (34)$$

corresponds to a solution to the \star -genvalue equation (27). Here, the L 's stand for the Laguerre polynomials defined by the Rodrigues formula

$$L_n(z) = \frac{e^z \partial^n (e^{-z} z^n)}{n!}. \quad (35)$$

As we will see in the next section, Wigner function (34) is intrinsically related to the Wigner function of the harmonic oscillator. Besides, it turns out that the Hamiltonian of the Pais–Uhlenbeck oscillator is canonically equivalent to the Hamiltonian corresponding to the difference of two uncoupled harmonic oscillators. This implies, as long as the canonical transformation remains linear [33], that the Wigner function of the Pais–Uhlenbeck is equivalent to the Wigner function of a pair of harmonic oscillators, but written in terms of its own Noether charges.

Also, after some algebraic manipulations, one may check by substituting the Wigner formula (34) into equation (29), that the system has energy \star -eigenvalues

$$E_{nm} = \left(n + \frac{1}{2}\right) \Omega_1 - \left(m + \frac{1}{2}\right) \Omega_2, \quad n, m = 0, 1, 2, \dots, \quad (36)$$

as it is well-known for the Pais–Uhlenbeck oscillator.

3.2. Wave function

In this section we are interested in obtaining the wave function associated to the quantum mechanical system of the Pais–Uhlenbeck oscillator within the framework of deformation quantization. To this end, we will use the formalism of quantum canonical transformations already discussed in section 2.2. This can be achieved by following the next strategy. First, by means of an appropriate canonical transformation, we will map the Hamiltonian of the Pais–Uhlenbeck oscillator into a Hamiltonian describing a pair of uncoupled harmonic oscillators. The second step will be to solve the Wigner function corresponding to the new Hamiltonian, and then, through a Fourier transformation, we will calculate the wave function related to the harmonic oscillator problem. Finally, we will consider a quantum canonical transformation in order to obtain the wave function belonging to the original problem, that is, the Pais–Uhlenbeck oscillator. The aim of this approach is twofold. On the one hand, instead of calculating the wave function directly from the Pais–Uhlenbeck Wigner distribution, we note that the quantum canonical transformation not only leads to more manageable integrals, but also it allows us to compare with previous results found in the literature where the wave function is obtained by solving the Schrödinger equation. On the other hand, the equal frequency limit $\Omega_1 \rightarrow \Omega_2$ is analyzed via canonical transformations at both, the classical and the quantum levels [3, 6].

Consider for definiteness, the case of unequal frequencies and assume $\Omega_1 > \Omega_2$. The Hamiltonian (24) can be brought into diagonal form by applying the canonical transformation [6],

$$\begin{aligned} q &= \frac{1}{\Omega_1} \frac{\Omega_1 X_2 - P_1}{\sqrt{\Omega_1^2 - \Omega_2^2}}, & x &= \frac{\Omega_1 X_1 - P_2}{\sqrt{\Omega_1^2 - \Omega_2^2}}, \\ p_x &= \frac{\Omega_1 P_1 - \Omega_2^2 X_2}{\sqrt{\Omega_1^2 - \Omega_2^2}}, & p_q &= \Omega_1 \frac{\Omega_1 P_2 - \Omega_2^2 X_1}{\sqrt{\Omega_1^2 - \Omega_2^2}}, \end{aligned} \quad (37)$$

which is realized by the generating function

$$F(q, x, X_1, X_2) = \Omega_1 \gamma q X_2 + \gamma x X_1 - \Omega_1^2 q x - \Omega_1 X_1 X_2, \quad (38)$$

where $\gamma := \sqrt{\Omega_1^2 - \Omega_2^2}$. Using such a canonical transformation, one finds that the Hamiltonian is mapped into the difference of an uncoupled pair of harmonic oscillators

$$H(X_1, P_1, X_2, P_2) = \frac{P_1^2 + \Omega_1^2 X_1^2}{2} - \frac{P_2^2 + \Omega_2^2 X_2^2}{2}. \quad (39)$$

Following the \star -genvalue equation (11) for this Hamiltonian, the resulting equation is

$$\begin{aligned} &\left[\left(P_1 - \frac{i\hbar}{2} \partial_{X_1} \right)^2 + \Omega_1^2 \left(X_1 + \frac{i\hbar}{2} \partial_{P_1} \right)^2 - \left(P_2 - \frac{i\hbar}{2} \partial_{X_2} \right)^2 \right. \\ &\quad \left. - \Omega_2^2 \left(X_2 + \frac{i\hbar}{2} \partial_{P_2} \right)^2 - 2E \right] \rho(X_1, X_2, P_1, P_2) = 0. \end{aligned} \quad (40)$$

By virtue of its imaginary part

$$\hbar (-P_1 \partial_{X_1} + \Omega_1^2 X_1 \partial_{P_1} + P_2 \partial_{X_2} - \Omega_2^2 X_2 \partial_{P_2}) \rho = 0, \quad (41)$$

the Wigner function ρ is seen to depend on two variables, $z_1 = 4H_1^{\text{osc}}/\hbar\Omega_1$ and $z_2 = 4H_2^{\text{osc}}/\hbar\Omega_2$, where $H_1^{\text{osc}} := (P_1^2 + \Omega_1^2 X_1^2)/2$ and $H_2^{\text{osc}} := (P_2^2 + \Omega_2^2 X_2^2)/2$ correspond to the Noether charges to the Hamiltonian of uncoupled harmonic oscillators (39). Then, the \star -genvalue equation, reduces to a couple of simple ordinary differential equations

$$\left(\frac{z_i}{4} - z_i \partial_{z_i}^2 - \partial_{z_i} - \frac{E}{2\hbar\Omega_i} \right) \rho(z_1, z_2) = 0, \quad i = 1, 2. \quad (42)$$

This equation acquires the same form as that for the simple harmonic oscillator [16], therefore, one may be easily convinced that the Wigner function is determined by

$$\rho_{nm}(X_1, P_1, X_2, P_2) = \frac{(-1)^{n+m}}{\pi^2 \hbar^2} e^{-\frac{2H_1^{\text{osc}}}{\hbar \Omega_1}} e^{-\frac{2H_2^{\text{osc}}}{\hbar \Omega_2}} L_n \left(\frac{4H_1^{\text{osc}}}{\hbar \Omega_1} \right) L_m \left(\frac{4H_2^{\text{osc}}}{\hbar \Omega_2} \right), \quad (43)$$

where the energy spectrum E results equal to the energy spectrum of the Pais–Uhlenbeck oscillator (36).

We can obtain the wave function $\psi(X_1, X_2)$ in the coordinates (X_1, X_2) by Fourier transforming in the momentum variables the Wigner function (43) adapted to our system, obtaining [40]

$$\psi(X_1, X_2) = \frac{1}{\psi^*(0, 0)} \int dP_1 dP_2 \rho \left(\frac{X_1}{2}, P_1, \frac{X_2}{2}, P_2 \right) e^{iP_1 X_1 + P_2 X_2 / \hbar}, \quad (44)$$

where the constant $\psi^*(0, 0)$ may be determined up to a phase by normalization of $\psi(X_1, X_2)$. By using the following identity between Laguerre and Hermite polynomials, denoted by H 's [41],

$$\int_{-\infty}^{\infty} dx \left[H_n(x - a) H_n(x + a) e^{-x^2} e^{-2ibx} \right] = 2^n \sqrt{\pi} n! e^{-b^2} L_n(2(a^2 + b^2)), \quad (45)$$

the integral gives the wave function in the position space

$$\psi_{nm}(X_1, X_2) = \frac{(-1)^{n+m}}{\pi^2 \hbar^2} e^{-\frac{\Omega_1^2 X_1^2}{2}} e^{-\frac{\Omega_2^2 X_2^2}{2}} H_n(\sqrt{\Omega_1} X_1) H_m(\sqrt{\Omega_2} X_2). \quad (46)$$

The final step consists in calculate the wave function associated to the Pais–Uhlenbeck oscillator. To this end, we make use of the quantum canonical transformation operator, defined in (19), where the generator, $F(q, x, X_1, X_2)$, of the classical canonical transformation takes the specific form of (38). Explicitly, the wave function for the Pais–Uhlenbeck oscillator in the position space is given by the formula

$$\begin{aligned} \psi_{nm}(q, x) &= N \int dX_1 dX_2 \left\{ \exp \left[\frac{i}{\hbar} (\Omega_1 \gamma q X_2 + \gamma x X_1 - \Omega_1^2 q x - \Omega_1 X_1 X_2) \right] \right. \\ &\quad \left. \psi_{nm}(X_1, X_2) \right\}, \end{aligned} \quad (47)$$

where N is a normalization constant. From now on, we will consider $\hbar = 1$ for simplicity. Also, here we only state our results, leaving the technical details

on the construction of the wave equation to the Appendix. Bearing this in mind, after performing the integration in (47), we are able to determine the wave function for the Pais–Uhlenbeck oscillator, which explicitly reads

$$\psi_{nm}(q, x) = N_{nm} \exp[-i\Omega_1\Omega_2qx] \exp\left[-\frac{\Delta}{2}(x^2 + \Omega_1\Omega_2q^2)\right] \phi_{nm}(q, x), \quad (48)$$

where $\Delta := \Omega_1 - \Omega_2$, and the functions ϕ_{nm} stand for

$$\begin{aligned} \phi_{nm}(q, x) &= \sum_{k=0}^m A_{\Delta}^k \frac{m!(n-m)!}{(m-k)!k!(n-m+k)!} H_{n-m+k}^+ H_k^-, \quad m \leq n, \\ \phi_{nm}(q, x) &= \sum_{k=0}^n A_{\Delta}^k \frac{n!(m-n)!}{(n-k)!k!(m-n+k)!} H_k^+ H_{m-n+k}^-, \quad m > n. \end{aligned} \quad (49)$$

Here, the constant $A_{\Delta} := i\Delta/4\sqrt{\Omega_1\Omega_2}$ only depends on the frequencies, while the H^+ 's and the H^- 's represent Hermite polynomial evaluated in the arguments

$$\begin{aligned} H_n^+ &:= H_n\left[i\sqrt{\Omega_1}(\Omega_2q - ix)\right], \\ H_n^- &:= H_n\left[i\sqrt{\Omega_2}(\Omega_1q + ix)\right], \end{aligned} \quad (50)$$

respectively, and the constants N_{nm} behave as normalization factors. The wave function (48) stand for the solution to the Schrödinger equation associated to the Hamiltonian operator of the Pais–Uhlenbeck oscillator [6]. As it was proven in [5], these solutions are normalized resulting in a pure point spectrum, and all eigenfunctions form a complete orthogonal basis in the Hilbert space $L^2(\mathbb{R}^2)$.

3.3. Equal frequency limit

In the equal frequency limit, $\Delta = \Omega_1 - \Omega_2 \rightarrow 0$, the quantum wave function (48) cease to be normalizable, implying that the spectrum acquires continuous values. Indeed, within our context the Wigner function (34) oscillates wildly but eventually approximating to zero, therefore, the probability distribution appears as a generalized function, in such a way that the energy of the system is equally likely to be found anywhere in an interval $[E, E + dE]$ for any E [42, 43]. Explicitly, whenever $\Delta = 0$ the source of the continuous

spectrum can be traced out through a quantum canonical transformation (19) defined via the linear classical canonical transformation

$$F(q, x, Q_1, Q_2) = \frac{qQ_2}{\sqrt{2}} - \frac{\Omega qx}{4} + \frac{\Omega xQ_1}{\sqrt{2}} - \frac{Q_1Q_2}{2}, \quad (51)$$

where we have considered the single frequency as $\Omega := \Omega_1 = \Omega_2$. This generating function maps the Pais–Uhlenbeck Hamiltonian (24) to a new Hamiltonian

$$H_\Omega = \Omega(Q_1P_2 - Q_2P_1) - \frac{\Omega^2}{4}(Q_1^2 + Q_2^2), \quad (52)$$

which clearly differs from that obtained as the difference of two uncoupled harmonic oscillators in (39). We also may easily deduce that the spectrum of the equal frequency Hamiltonian (52) is composed of a discrete spectrum coming from the angular momentum part $Q_1P_2 - Q_2P_1$, and of a continuous spectrum originated from the squared norm of the position variables $Q_1^2 + Q_2^2$ as demonstrated in Ref. [4],

$$E_{mk} = \Omega\hbar \left(m - \frac{\Omega\hbar k^2}{4} \right). \quad (53)$$

Moreover, as it was indicated in [6, 4], the Pais–Uhlenbeck system results ghost-free, as even in the equal frequency limit $\Delta \rightarrow 0$ the evolution operator is certainly unitary. Again, within our context, the unitary property is readily obtained from the Wigner functions (34) and (43), as both are related by a quantum canonical transformation of the form (18), where the unitary operator \widehat{U}_T is in fact given by the integral transformation (47). Following this reasoning, since the time evolution of the pair of harmonic oscillators is a solution of Moyal’s equation [16]

$$\rho(X_1, X_2, P_1, P_2, t) = U_\star^{-1} \star \rho(X_1, X_2, P_1, P_2, 0) \star U_\star, \quad (54)$$

where the \star -evolution operator

$$\begin{aligned} U_\star(X_1, X_2, P_1, P_2) &= e_\star^{itH/\hbar} := 1 + \left(\frac{it}{\hbar} \right) H + \frac{1}{2!} \left(\frac{it}{\hbar} \right)^2 H \star H + \dots \\ &= 2\pi\hbar \sum_{n,m} e^{itE_{nm}/\hbar} \rho_{nm} \end{aligned} \quad (55)$$

corresponds to a unitary operator. Therefore, the evolution of the Pais–Uhlenbeck oscillator is also unitary, as a consequence.

4. Discussion

In this article, we analyzed the fourth order Pais–Uhlenbeck model within the Deformation quantization formalism. We obtained both, the Wigner distribution function and the wave equation, for this system. In particular, in order to obtain the Wigner function we explicitly consider the symmetries of the system associated to its Noether charges which were established as ad hoc variables for solving the \star -genvalue equation. For the wave equation, we proceed by first considering a classical canonical transformation that map the Pais–Uhlenbeck model to a system of uncoupled harmonic oscillators. Afterward, our strategy was to obtain the Wigner function for the new system, and then to construct the associated wave function for this system. Finally, we reached the wave function for the Pais–Uhlenbeck model by a quantum canonical transformation. This wave equation resulted identical to the one considered in [6]. We also showed that the model contains a continuous spectrum and results ghost-free, the two conditions together being a consequence of the unitariness of the relevant quantum canonical transformations.

Further studies are necessary in order to see the generality of conserved quantities associated to Noether charges as appropriate variables to find solutions to the \star -genvalue equation for a generic model. This will be done elsewhere.

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Appendix A. Technical steps for the construction of the Pais–Uhlenbeck wave function

In the following we will detail some calculations that were omitted in the main text. Explicitly, by means of the quantum canonical transformation (47), the wave function for the Pais–Uhlenbeck system (47) can be expressed as

$$\psi(q, x) = \exp[-i\Omega_1^2 qx] \int dX_1 dX_2 \exp\left[i\gamma x X_2 - \frac{\Omega_2 X_2^2}{2} + i\Omega_1 \gamma q X_1\right]$$

$$\left. -i\Omega_1 X_1 X_2 - \frac{\Omega_1 X_1^2}{2} \right] H_m \left(\sqrt{\Omega_1} X_1 \right) H_m \left(\sqrt{\Omega_2} X_2 \right). \quad (\text{A.1})$$

Defining $Z_1 := \sqrt{\Omega_1} X_1$ and $Z_2 := \sqrt{\Omega_2} X_2$, this expression may be written as

$$\begin{aligned} \psi(q, x) &= \frac{\exp[-i\Omega_1^2 q x]}{\sqrt{\Omega_1 \Omega_2}} \int dZ_1 dZ_2 \exp \left[\frac{i\gamma x Z_2}{\sqrt{\Omega_1}} - \frac{Z_2^2}{2} + i\sqrt{\Omega_1} \gamma q Z_1 \right. \\ &\quad \left. - \sqrt{\frac{\Omega_1}{\Omega_2}} Z_1 Z_2 - \frac{Z_1^2}{2} \right] H_n(Z_1) H_m(Z_2). \end{aligned} \quad (\text{A.2})$$

Multiplying both sides by $\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{t^n}{n!} \frac{u^m}{m!}$, and using the generating function for the Hermite polynomials [41]

$$\exp(-t^2 + 2tx) = \sum_{k=0}^{\infty} \frac{t^k}{k!} H_k(x), \quad (\text{A.3})$$

we perform the Z_1 and Z_2 integration using the Gaussian integral [41]

$$\int_{-\infty}^{\infty} dx \exp(-p^2 x^2 + qx) = \exp\left(\frac{q^2}{4p^2} \frac{\sqrt{\pi}}{p}\right), \quad (\text{A.4})$$

thus obtaining in this way the identity

$$\begin{aligned} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{t^k}{k!} \frac{u^l}{l!} \psi(q, x) &= 2\pi \exp[-i\Omega_1 \Omega_2 q x] \exp\left[-\frac{\Delta}{2} (x^2 + \Omega_1 \Omega_2 q^2)\right] \\ &\quad \times \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \sum_{j=0}^{\infty} \left\{ \left(-4i\sqrt{\Omega_1 \Omega_2}\right)^j \left(\frac{(\Omega_1 - \Omega_2)^{(k+l)/2}}{(\Omega_1 + \Omega_2)^{(k+l+2j)/2}}\right) \right. \\ &\quad \left. \times \left(\frac{t^{k+j} u^{l+j}}{k! l! j!}\right) H_k(x^+) H_l(x^-) \right\}. \end{aligned} \quad (\text{A.5})$$

Using the double summation identities [44],

$$\begin{aligned} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} A_{k,n} &= \sum_{n=0}^{\infty} \sum_{k=0}^n A_{k,n-k}, \\ \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} A_{k,n} &= \sum_{n=0}^{\infty} \sum_{k=0}^n A_{k,n}, \end{aligned} \quad (\text{A.6})$$

and relabeling the summation indices, we finally obtain that the n, m -term of the sum (A.5), defined as ψ_{nm} , results identical to the expression (48).

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